Towards Understanding Triangle Construction Problems

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Geometry Construction Problems in Mathematics

- One of the longest, constantly studied problems in mathematics and mathematical education (for more than 2500 years); also, some applications in CAD
- Goal: construct a geometry figure that meets given constraints
- Constructions are procedures (over a suitable language)
- Some instances are unsolvable (e.g. angle trisection, cube doubling,...)
- General problem is decidable, but algebraic-style solutions are not always suitable

Solutions of Construction Problems

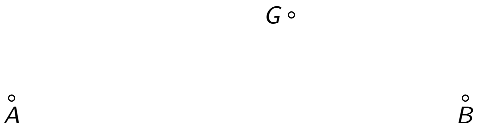
Components of solutions of construction problems:

- Analysis: finding properties that enable a construction
- Construction: a concrete construction procedure
- Proof: the constructed figure meets the given specification
- Discussion: how many possible solutions there are and under what conditions

Constructions with Straightedge and Compass

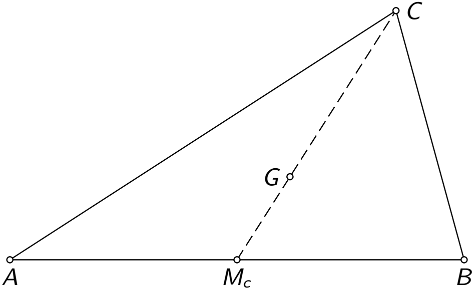
- Tools: straightedge (not ruler) and collapsible compass
- Typically used: construction steps compound from elementary construction steps (e.g., construct the segment midpoint)
- Main obstacle: combinatorial explosion — huge search space:
  - many different construction steps available
  - plenty of objects that each step could be applied to
- We focus on triangle construction problems

Example Problem



Problem: *Construct a triangle ABC given vertices A and B and the barycenter G*

Example Solution



Construction: *Construct the midpoint* M_c *of the segment* AB; *then construct the vertex* C *such that* $M_cG : M_cC = 1/3$

Existing Approaches and Corpora

- Several existing approaches, including:
  - Schreck (1995)
  - Gao and Chou (1998)
  - Gulwani et al. (2011)

Wernick's Corpus

- One of systematically built corpora, created in 1982, some variants in the meanwhile
- Task: construct a triangle given three located points selected from the following list:
  - A, B, C – vertices
  - I, O – incenter and circumcenter
  - H, G – orthocenter and barycenter
  - M_a, M_b, M_c – the side midpoints
  - H_a, H_b, H_c – feet of altitudes
  - T_a, T_b, T_c – intersections of the internal angles bisectors with the opposite sides

Wernick's Problems (2)

139 non-trivial, significantly different, problems; 25 redundant (R) or locus-restricted (L); 72 solvable (S), 16 unsolvable (U); 25 still with unknown status

1. A, B, O L		57. A, H, I S [9]	85. M_a, M_b, H_a S	113. M_a, T_b, T_c
2. A, B, M_a S		A, T_a, T_b S [9]	86. M_a, M_b, H_c S	114. M_a, T_b, I U [9]
3. A, B, M_c R		T_a, I L	87. M_a, M_b, H S [9]	115. G, H_a, H_b U [9]
4. A, B, G S		T_b, T_c S	88. M_a, M_b, T_a U [9]	116. G, H_a, H S
5. A, B, H_a L		$, I$ S	89. M_a, M_b, T_c U [9]	117. G, H_a, T_a S
6. A, B, H_c L		M_b S	90. M_a, M_b, I U [10]	118. G, H_a, T_b
7. A, B, H S		G S	91. M_a, G, H_a L	119. G, H_a, I
8. A, B, T_a S		H_a L	92. M_a, G, H_b S	120. G, H, T_a U [9]
9. A, B, T_c		T_b S	93. M_a, G, H S	121. G, H, I U [9]
		S	94. M_a, G, T_a S	122. G, T_a, T_b
		L	95. M_a, G, T_b U [9]	123. G, T_a, I
		U [9]	96. M_a, G, I S [9]	124. H_a, H_b, H_c S
		S	97. M_a, H_a, H_b S	125. H_a, H_b, H S
		S	98. M_a, H_a, H L	126. H_a, H_b, T_a S
		R	99. M_a, H_a, T_a L	127. H_a, H_b, T_c
		U [9]	100. M_a, H_a, T_b U [9]	128. H_a, H_b, I
		U [9]	101. M_a, H_a, I S	129. H_a, H, T_a L
		H_b U [9]	102. M_a, H_b, H_c L	130. H_a, H, T_b U [9]
		$, H$ S	103. M_a, H_b, H S	131. H_a, H, I S [9]
		$, T_a$ S	104. M_a, H_b, T_a S	132. H_a, T_a, T_b
		H_a, T_b	105. M_a, H_b, T_b S	133. H_a, T_a, I S
		$, H_a, I$	106. M_a, H_b, T_c U [9]	134. H_a, T_b, T_c
		O, H, T_a U [9]	107. M_a, H_b, I U [9]	135. H_a, T_b, I
		80. O, H, I U [9]	108. M_a, H, T_a U [9]	136. H, T_a, T_b
25. $A,$		81. O, T_a, T_b	109. M_a, H, T_b U [10]	137. H, T_a, I
26. $A, M_a,$	$, I$ S	82. O, T_a, I S [9]	110. M_a, H, I U [10]	138. T_a, T_b, T_c U [11]
27. A, M_a, I S [9]	55. A, H, T_a S	83. M_a, M_b, M_c S	111. M_a, T_a, T_b U [10]	139. T_a, T_b, I S
28. A, M_b, M_c S	56. A, H, T_b U [9]	84. M_a, M_b, G S	112. M_a, T_a, I S	

Basic Approach (1)

- A careful analysis of all available solutions performed
- Solutions use high-level rules, e.g:
  - *if barycenter G and circumcenter O are known, then the orthocenter H can be constructed*
  - *if two triangle vertices are given, then the side bisector can be constructed*
- In total: $\approx$ 70 rules used

Basic Approach (2)

- Implemented in Prolog
- Simple forward chaining mechanism for search procedure
- Solves most of solvable examples from Wernick's list in less than 1s and with the maximal search depth 9
- But... there are too many rules! (it is not problem to search over them, but to invent and systematize them)

Separation of Concepts – Definitions, Lemmas, Construction Steps (1)

Motivating example: *Construct the midpoint M_c of AB and then construct C such that $M_cG : M_cC = 1 : 3$* uses the following:

- M_c is the side midpoint of AB
- G is the barycenter of ABC
- it holds that $M_cG = 1/3M_cC$
- given points X and Y, it is possible to construct the midpoint of the segment XY
- given points X and Y, it is possible to construct a point Z, such that: $XY : XZ = 1 : k$

Separation of Concepts – Definitions, Lemmas, Construction Steps (2)

Motivating example: *Construct the midpoint M_c of AB and then construct C such that $M_cG : M_cC = 1 : 3$* uses the following:

- M_c is the side midpoint of AB (definition of M_c)
- G is the barycenter of ABC (definition of G)
- it holds that $M_cG = 1/3M_cC$ (lemma)
- given points X and Y, it is possible to construct the midpoint of the segment XY (construction primitive)
- given points X and Y, it is possible to construct a point Z, such that: $XY : XZ = 1 : k$ (construction primitive)

Advanced Approach

- Task: Determine the sets of definitions, lemmas and construction primitives such that all needed high-level (instantiated) construction rules can be built from them
- From:
  - it holds that $M_c G = 1/3 M_c C$ (lemma)
  - given points X and Y, it is possible to construct a point Z, such that: $XY : XZ = 1 : r$ (construction primitive)

  we can derive:
  - given M_c and G, it is possible to construct C

Advanced Approach: Rule Derivation

- Controlled instantiations of lemmas
- All construction rules derived from:
  - 11 definitions (including Wernick's notation)
  - 29 simple lemmas
  - 18 construction primitives (including elementary construction steps)
- Deriving rules is performed once, in preprocessing phase (takes approx. 20s)

Advanced Approach: Re-evaluation

- Another corpus: construct a triangle given three lengths from the following set:
  - $|AB|$, $|BC|$, $|AC|$: lengths of the sides;
  - $|AM_a|$, $|BM_b|$, $|CM_c|$: lengths of the medians;
  - $|AH_a|$, $|BH_b|$, $|CH_c|$: lengths of the altitudes.
- For 17 (out of total of 20) problems, additional: 2 defs, 2 lemmas, and 9 construction steps were needed
- For additional corpora, we expect less and less additions

Output: Constructions in a Natural Language Form (Example)

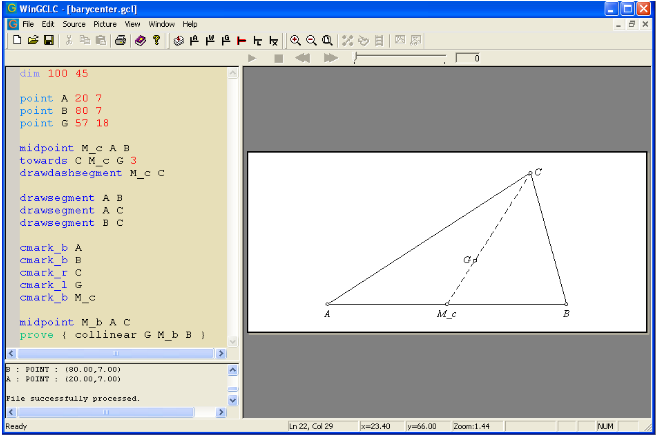
Generated construction for the problem 53 $(A; H_b; T_c)$:

1. Using A and H_b, construct the line AC;
2. Using A and T_c, construct the line AB;
3. Using H_b and AC, construct the line BH_b;
4. Using AB and BH_b, construct the point B;
5. Using A and B and T_c, construct the point T_c';
6. Using T_c and T_c', construct the circle over $T_c T_c'$;
7. Using circle over $T_c T_c'$ and AC, construct the point C.

Output: Constructions in GCLC Form (Example)

```
% free points
point A 30 5
point B 70 5
point G 57 14
% synthesized construction
midpoint M_c A B
towards C M_c G 3
drawdashsegment M_c C
% drawing the triangle ABC
drawsegment A B
drawsegment A C
drawsegment B C
```

Output: Constructions in GCLC Form (Example) (2)



Verification

- But... it is not only about synthesis/constructing!
- Verification (correctness proof) is also needed (not "correct by construction")
- "If the objects ... are constructed in the given way, then they meet the specification"
- Geometry theorem provers can be used (e.g. the area method, the Gröbner bases method, Wu's method)
- Again within GCLC tool
- The prover also provide NDG conditions

Discussion

1. But... it is not only about synthesis and verification!
2. Do the constructed objects exist at all? (recall: "If the objects ... are constructed in the given way, then they meet the specification")
3. Using the NDG conditions provided by the provers, we should prove that the constructed objects do exist
4. For this task we are planning to use our prover for coherent logic and generate formal proofs

Current and Future Work

- We are planning to
  - automatically produce formal proofs (in Isabelle) that the constructed objects do exist
  - prove correctness of generated constructions by using theorem provers from proof assistants
- We are planning to cover all corpora of triangle construction problems from the literature
- We are planning to automatically prove/derive all lemmas/construction rules from axioms/elementary construction steps

Conclusions

- First steps towards formally established solving of large collections of construction problems
- Product: a solver and a systematization of relevant definitions/lemmas/construction steps
- Aiming at covering all corpora from the literature (completeness claimed w.r.t. certain corpus)