Verifying an algorithm computing Discrete Vector Fields for digital imaging*

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Calculemus 2012

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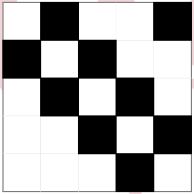
Algebraic Topology and Digital Images

Digital Image



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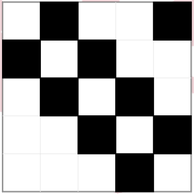

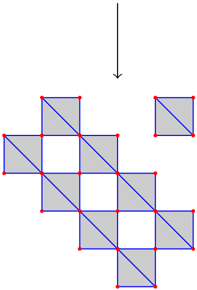

Simplicial complex

Algebraic Topology and Digital Images

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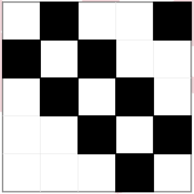

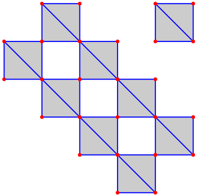

Simplicial complex

$$C_0 = \mathbb{Z}_2[vertices]$$
$$C_1 = \mathbb{Z}_2[edges]$$
$$C_2 = \mathbb{Z}_2[triangles]$$

Chain complex

Algebraic Topology and Digital Images

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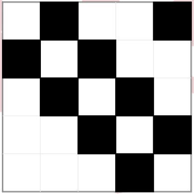

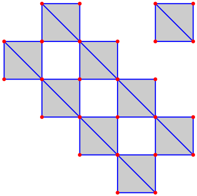

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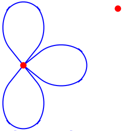

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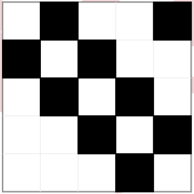

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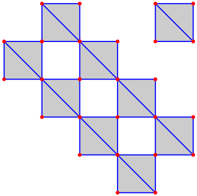

Homology groups

$$H_0 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$
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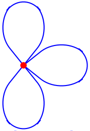

Reduced Chain Complex

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Chain complex

Algebraic Topology and Digital Images

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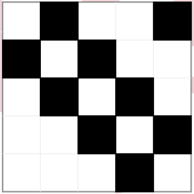

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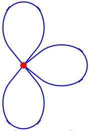

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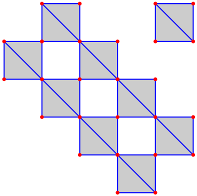

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- Application:
  - Analysis of biomedical images

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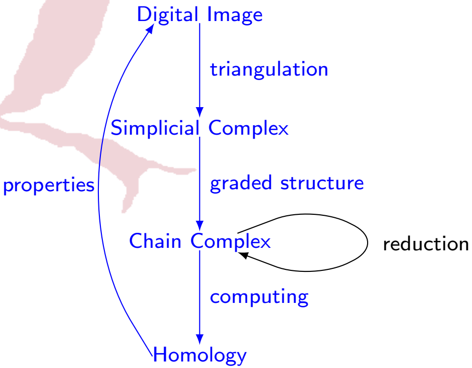
A formally verified efficient method to compute homology from digital images

Goal

$$
\begin{tikzcd}
\text{Digital Image} \arrow[d, "\text{triangulation}"] \\
\text{Simplicial Complex} \arrow[d, "\text{graded structure}"] \\
\text{Chain Complex} \arrow[d, "\text{computing}"] \arrow[loop right, "\text{reduction}"] \\
\text{Homology} \arrow[uuu, bend left, "\text{properties}"]
\end{tikzcd}
$$

Goal



J. Heras, M. Dénès, G. Mata, A. Mörtberg, M. Poza, and V. Siles. Towards a certified computation of homology groups. In proceedings 4th International Workshop on Computational Topology in Image Context. Lecture Notes in Computer Science, 7309, pages 49–57, 2012.

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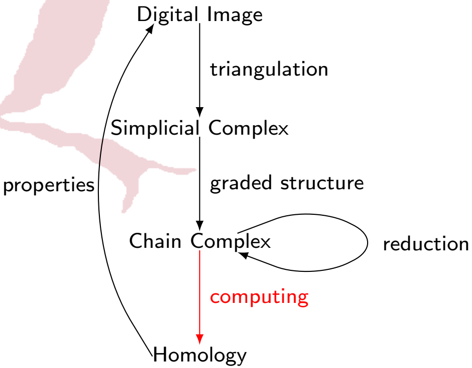
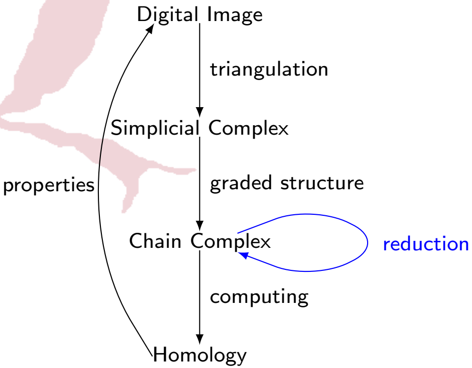


Bottleneck

Compute Homology from Chain Complexes

Goal



Goal of this work

Formalization in Coq/SSReflect of a procedure to reduce the size of Chain Complexes but preserving homology

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Chain Complexes

Definition

A chain complex C_* *is a pair of sequences* $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ *where:*

- *For every* $q \in \mathbb{Z}$, *the component* C_q *is a* $\mathbb{Z}_2$*-module, the chain group of degree* q
- *For every* $q \in \mathbb{Z}$, *the component* d_q *is a module morphism* $d_q : C_q \to C_{q-1}$, *the differential map*
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Definition

If $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ *is a chain complex:*

- *The image* $B_q = im\ d_{q+1} \subseteq C_q$ *is the (sub)module of q-boundaries*
- *The kernel* $Z_q = ker\ d_q \subseteq C_q$ *is the (sub)module of q-cycles*

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Definition

Let $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ *be a chain complex. For each degree* $n \in \mathbb{Z}$*, the n-homology module of* C_* *is defined as the quotient module*

$$H_n(C_*) = \frac{Z_n}{B_n}$$

Reduction

Definition

A reduction ρ between two chain complexes C_* y D_* (denoted by $\rho : C_* \Rightarrow D_*$) is a tern $\rho = (f, g, h)$

$$\begin{tikzcd} C_* \arrow[loop left, "h"] \arrow[r, bend left, "f"] & D_* \arrow[l, bend left, "g"] \end{tikzcd}$$

satisfying the following relations:

1) $fg = id_{D_*}$;
2) $d_C h + hd_C = id_{C_*} - gf$;
3) $fh = 0; \quad hg = 0; \quad hh = 0.$

Theorem

If $C_* \Rightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, what implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

Discrete Morse Theory

A. Romero and F. Sergeraert. Discrete Vector Fields and Fundamental Algebraic Topology, 2010. http://arxiv.org/abs/1005.5685v1.

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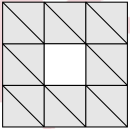

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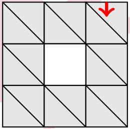

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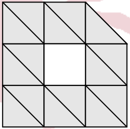

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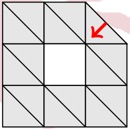

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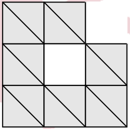

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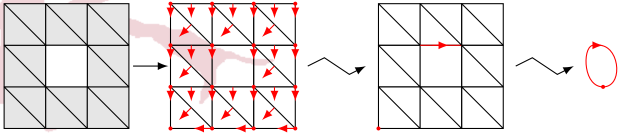

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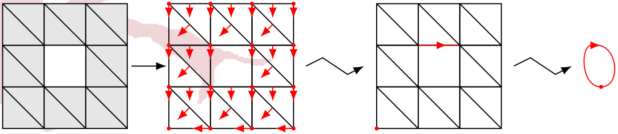

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  - $C_* \Rightarrow C_*^c$
  - generators of C_*^c are critical cells of C_*

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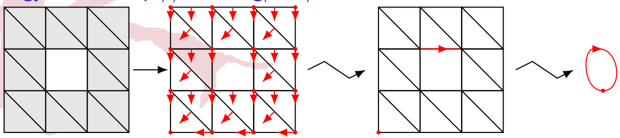

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$$\begin{array}{ccccccccc} 0 & \leftarrow & \mathbb{Z}_2^{16} & \xleftarrow{d_1} & \mathbb{Z}_2^{32} & \xleftarrow{d_2} & \mathbb{Z}_2^{16} & \leftarrow & 0 \\ & & & & \downarrow & & & & \\ 0 & \leftarrow & \mathbb{Z}_2 & \xleftarrow{\widehat{d_1}} & \mathbb{Z}_2 & \xleftarrow{\widehat{d_2}} & 0 & \leftarrow & 0 \end{array}$$

Discrete Vector Fields

Definition

Let $C_* = (C_p, d_p)_{p\in\mathbb{Z}}$ a free chain complex with distinguished $\mathbb{Z}_2$-basis $\beta_p \subset C_p$. A discrete vector field V on C_* is a collection of pairs $V = \{(\sigma_i; \tau_i)\}_{i\in I}$ satisfying the conditions:

- *Every σ_i is some element of β_p, in which case $\tau_i \in \beta_{p+1}$. The degree p depends on i and in general is not constant.*
- *Every component σ_i is a regular face of the corresponding τ_i.*
- *Each generator (cell) of C_* appears at most once in V.*

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Definition

A V-path of degree p and length m is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0\leq k<m}$ satisfying:

- *Every pair $(\sigma_{i_k}, \tau_{i_k})$ is a component of V and τ_{i_k} is a p-cell.*
- *For every $0 < k < m$, the component σ_{i_k} is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$.*

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A discrete vector field V is admissible if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every V-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

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A cell σ which does not appear in a discrete vector field V is called a critical cell.

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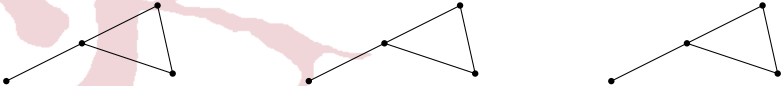
A cell σ which does not appear in a discrete vector field V is called a critical cell.

Theorem

Let $C_* = (C_p, d_p)_{p\in\mathbb{Z}}$ be a free chain complex and $V = \{(\sigma_i;\tau_i)\}_{i\in I}$ be an admissible discrete vector field on C_*. Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow (C_p^c, d_p')$ where $C_p^c = \mathbb{Z}_2[\beta_p^c]$ is the free $\mathbb{Z}_2$-module generated by the critical p-cells.

Example: an admissible discrete vector field



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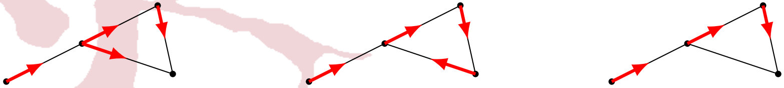

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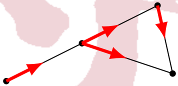

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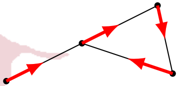

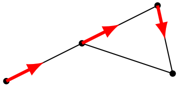

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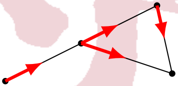

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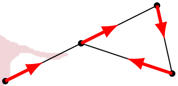

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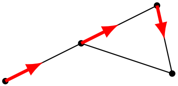

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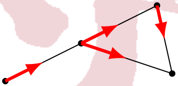

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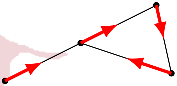

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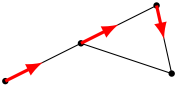

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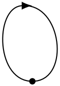

Vector fields and integer matrices

Differential maps of a Chain Complex can be represented as matrices

$$\ldots \leftarrow \mathbb{Z}_2^m \xleftarrow{M} \mathbb{Z}_2^n \leftarrow \ldots$$

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An admissible vector field V for M is nothing but a set of integer pairs $\{(a_i, b_i)\}$ satisfying these conditions:

1. $1 \leq a_i \leq m$ *and* $1 \leq b_i \leq n$
2. *The entry* $M[a_i, b_i]$ *of the matrix is* 1
3. *The indices* a_i *(resp.* b_i*) are pairwise different*
4. *Non existence of loops*

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Coq/SSReflect

- Coq:
  - An Interactive Proof Assistant
  - Based on Calculus of Inductive Constructions
  - Interesting feature: program extraction from a constructive proof

Coq/SSReflect

- Coq:
  - An Interactive Proof Assistant
  - Based on Calculus of Inductive Constructions
  - Interesting feature: program extraction from a constructive proof
- SSReflect:
  - Extension of Coq
  - Developed while formalizing the Four Color Theorem by G. Gonthier
  - Currently, it is used in the formalization of Feit-Thompson Theorem

Admissible discrete vector fields in SSReflect

Definition

An admissible discrete vector field V for M is nothing but a set of integer pairs $\{(a_i, b_i)\}$ satisfying these conditions:

1. $1 \leq a_i \leq m$ *and* $1 \leq b_i \leq n$
2. *The entry* $M[a_i, b_i]$ *of the matrix is* 1
3. *The indices* a_i *(resp.* b_i*) are pairwise different*
4. *Non existence of loops*

```
Definition admissible_dvf (M: 'M[Z2]_(m,n))
                  (V: seq ('I_m * 'I_n)) (ords : simpl_rel 'I_m) :=
  all [pred p | M p.1 p.2 == 1] V &&
  uniq (map (@fst _ _) V) && uniq (map (@snd _ _) V) &&
  all [pred i | ~~ (connect ords i i)] (map (@fst _ _) V).
```

The abstract algorithm

```
Fixpoint genDvfOrders M V (ords : simpl_rel _) k :=
  if k is l.+1 then
    let P := [pred ij | admissible (ij::V) M
                         (relU ords (gen_orders M ij.1 ij.2))] in
    if pick P is Some (i,j)
       then genDvfOrders M ((i,j)::V)
                          (relU ords (gen_orders M i j)) l
    else (V, ords)
  else (V, ords).

Definition gen_adm_dvf M :=
  genDvfOrders M [::] [rel x y | false] (minn m n).
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The abstract algorithm

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Fixpoint genDvfOrders M V (ords : simpl_rel _) k :=
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Definition gen_adm_dvf M :=
  genDvfOrders M [::] [rel x y | false] (minn m n).

Lemma admissible_gen_adm_dvf m n (M : 'M[Z2]_(m,n)) :
 let (vf,ords) := gen_adm_dvf M in admissible vf M ords.
```

Problem

It is not an executable algorithm

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Romero-Sergeraert's Algorithm

A. Romero and F. Sergeraert. Discrete Vector Fields and Fundamental Algebraic Topology, 2010. http://arxiv.org/abs/1005.5685v1.

Algorithm

Input: A matrix M
Output: An admissible discrete vector field for M

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Algorithm

Input: A matrix M
Output: An admissible discrete vector field for M

Algorithm

Input: A chain complex C_* *and an admissible discrete vector field of* C_*
Output: A reduced chain complex $\hat{C}_*$

From computation to verification through testing

- *Haskell* as programming language
- *QuickCheck* to test the programs
- *Coq/SSReflect* to verify the correctness of the programs

Haskell

Algorithm (*gen_adm_dvf*)

Input: A matrix M
Output: An admissible discrete vector field for M

Algorithm (*reduced_cc*)

Input: A chain complex C_*
Output: A reduced chain complex $\hat{C}_*$

```
> gen_adm_dvf [[1,0,1,1],[0,0,1,0],[1,1,0,1]]
[(0,0),(1,2),(2,1)]
```

QuickCheck

- A specification of the properties which our program must verify
- Testing them
  - Towards verification
  - Detect bugs

```
> quickCheck M -> admissible (gen_adm_dvf M)
+ + + OK, passed 100 tests
```

Coq/SSReflect

SSReflect Theorem:

```
Theorem gen_adm_dvf_is_admissible (M : seq (seq Z2)) :
 admissible (gen_adm_dvf M).
```

SSReflect Theorem:

```
Theorem is_reduction (C : chaincomplex) : reduction C (reduced_cc C).
```

SSReflect Theorem:

```
Theorem reduction_preserves_betti (C D : chaincomplex)
  (rho : reduction C D) : Betti C = Betti D.
```

Experimental results

500 randomly generated matrices

- Initial size of the matrices: 100×300
- Time: 12 seconds

Experimental results

500 randomly generated matrices

- Initial size of the matrices: 100×300
- Time: 12 seconds

- After reduction: 5×50
- Time: milliseconds

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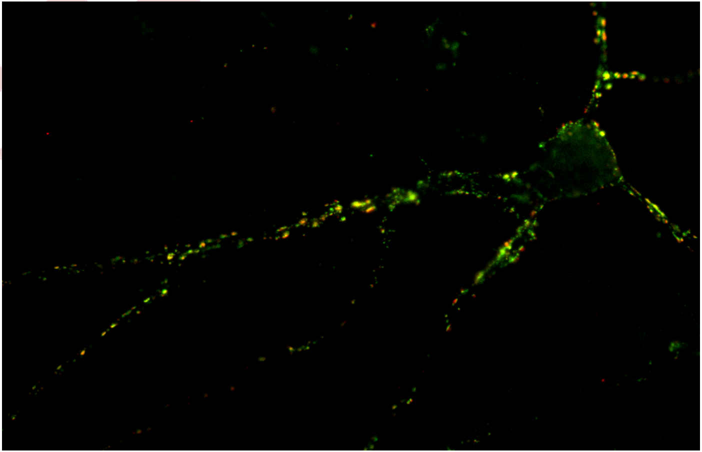
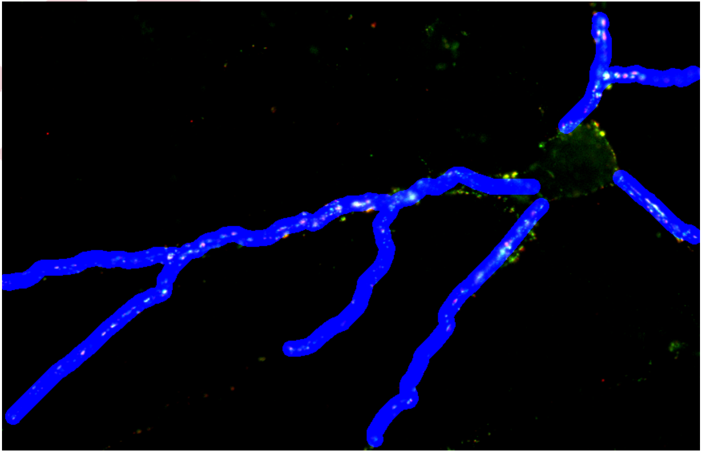
Counting Synapses

- *Synapses* are the points of connection between neurons
- *Relevance*: Computational capabilities of the brain
- Procedures to modify the synaptic density may be an important asset in the treatment of neurological diseases
- An automated and reliable method is necessary

Counting Synapses



Counting Synapses



Counting Synapses

Results with biomedical images

- Without reduction procedure:
  - Coq is not able to compute homology of this kind of images

Results with biomedical images

- Without reduction procedure:
  - Coq is not able to compute homology of this kind of images
- After reduction procedure:
  - Coq computes in just 25 seconds

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Conclusions and Further work

- Conclusions:
  - Method to reduce big images preserving homology
  - Formalization of admissible discrete vector fields on Coq
  - Remarkable reductions in different benchmarks

Conclusions and Further work

- Conclusions:
  - Method to reduce big images preserving homology
  - Formalization of admissible discrete vector fields on Coq
  - Remarkable reductions in different benchmarks
- Further work:
  - Matrices with coefficients over $\mathbb{Z}$
  - Integration between Coq and ACL2
  - Application of homological methods to biomedical problems

Verifying an algorithm computing Discrete Vector Fields for digital imaging

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Calculemus 2012